S^1-valued Sobolev maps

Petru Mironescu

$\mathbb{S}^1$-VALUED SOBOLEV MAPS

PETRU MIRONESCU

Abstract. We describe the structure of the space $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$, where $0 < s < \infty$ and $1 \le p < \infty$. According to the values of s, p and n, maps in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ can either be characterised by their phases, or by a couple (singular set, phase).
Here are two examples:

$$W^{1/2,6}(\mathbb{S}^3;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{1/2,6} + W^{1,3}\};$$

$$W^{1/2,3}(\mathbb{S}^2;\mathbb{S}^1) \approx D \times \{e^{\imath\varphi}\ ;\ \varphi \in W^{1/2,3} + W^{1,3/2}\}.$$

In the second example, D is an appropriate set of infinite sums of Dirac masses. The sense of $\approx$ will be explained in the paper.
The presentation is based on papers of H.-M. Nguyen [22], of the author [20] and on a joint forthcoming paper of H. Brezis, H.-M. Nguyen and the author [15].
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Contents

1. Introduction

In this paper, we will describe all the maps in the space

$$X = X_{s,p} = W^{s,p}(\mathbb{S}^n;\mathbb{S}^1) := \{u : \mathbb{S}^n \to \mathbb{S}^1\ ;\ u \in W^{s,p}\}.$$

Here, $n \ge 2$, $0 < s < \infty$ and $1 \le p < \infty$.
By analogy with the case of continuous (or C^k) $\mathbb{S}^1$-valued maps on $\mathbb{S}^n$, which are precisely the maps of the form $u = e^{\imath\varphi}$, with φ real and continuous (or C^k), the first guess is

$$X = e^{\imath Y}, \qquad \text{where } Y = Y_{s,p} := W^{s,p}(\mathbb{S}^n;\mathbb{R}). \tag{1.1}$$

Equality 1.1 was studied in [7]. [7] combined with a subsequent result [13] yields the following

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Theorem 1.1. *Let $0 < s < \infty$, $1 \le p < \infty$ and $n \ge 2$. Then*

(1) *for $sp < 1$ or $sp \ge n$, we have*
$$W^{s,p}(\mathbb{S}^n;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p}(\mathbb{S}^n;\mathbb{R})\}$$

(2) *for $n \ge 3$, $s \ge 1$, $2 \le sp < n$, we have*
$$W^{s,p}(\mathbb{S}^n;\mathbb{S}^1) = \{e^{\imath\varphi}\ ;\ \varphi \in W^{s,p} \cap W^{1,sp}(\mathbb{S}^n;\mathbb{R})\}.$$

Thus, in each of these cases, $\mathbb{S}^1$-valued maps with $W^{s,p}$-regularity have phases with (at least) $W^{s,p}$-regularity.

Theorem 1.2. *In the remaining cases, i. e.*

(1) *$0 < s < 1$, $1 \le sp < n$*
(2) *$s \ge 1$, $1 \le sp < 2$,*

$\mathbb{S}^1$-valued maps with $W^{s,p}$-regularity need not have $W^{s,p}$ phases.

In what follows, we address the question of the description of $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ in the cases left open by Theorem 1.2.

We start by giving two basic examples of $W^{s,p}$-maps that do not have $W^{s,p}$-phases.

"Analytical" example. The construction relies on three ingredients: optimality of the Sobolev embeddings, Gagliardo-Nirenberg inequalities and a uniqueness argument (Lemma 1.2).
We start by recalling

Lemma 1.1. *(Gagliardo-Nirenberg inequalities) For $0 < s < \infty$, $1 \le p < \infty$, $0 < t < 1$, we have $W^{s,p} \cap L^\infty \subset W^{ts,p/t}$.*
Exception: these inclusions do not hold in the exceptional case $s = 1$, $p = 1$, i. e., we do not have $W^{1,1} \cap L^\infty \subset W^{t,1/t}$.

Lemma 1.2. [7] *Let $s_j > 0$, $p_j \ge 1$ be such that $s_j p_j \ge 1$, $j = 1, \ldots, m$. Let $\varphi : \mathbb{S}^n \to 2\pi\mathbb{Z}$ be such that $\varphi \in \sum_{j=1}^{m} W^{s_j,p_j}$. Then φ is constant.*

We may now construct the analytical example. Assume that $0 < s < 1$ and $1 < sp < n$. Let $\psi \in W^{1,sp} \backslash W^{s,p}$ (such a ψ exists, by the Sobolev "non embedding" $W^{1,sp} \not\subset W^{s,p}$ for $0 < s < 1$ and $1 < sp < n$). Let $u := e^{\imath\psi}$. Then $u \in W^{1,sp}$ (since $\psi \in W^{1,sp}$). Since we also have $u \in L^\infty$, we have $u \in W^{s,p}$ (by Lemma 1.1). We claim that u has no lifting in $\varphi \in W^{s,p}$. Argue by contradiction: otherwise, we find that $\varphi - \psi$ is in $W^{1,sp} + W^{s,p}$ and is $2\pi\mathbb{Z}$-valued. Therefore, $\varphi - \psi$ is constant (Lemma 1.2). This implies that $\psi \in W^{s,p}$, a contradiction.

The above argument does not hold in the case $0 < s < 1$, $sp = 1$, since in that case we do not have $W^{1,sp} \cap L^\infty \subset W^{s,p}$ anymore. However, we may still find some $\varphi \in W^{1,1} \setminus W^{s,p}$ such that $e^{\imath\varphi} \in W^{s,p}$ and obtain, as above, that such u does not lift in $W^{s,p}$.

"Topological" example. We will consider maps defined on the unit ball B of $\mathbb{R}^n$. (It is easy to adapt the construction below for maps on $\mathbb{S}^n$.) Let $u : B \to \mathbb{S}^1$, $u(x) = \dfrac{(x_1, x_2)}{|(x_1, x_2)|}$. One can check that $u \in W^{s,p}$ whenever $sp < 2$.

We claim that u has no lifting in $W^{s,p}$ provided $1 < sp < 2$. Argue by contradiction and assume that $u = e^{\imath\varphi}$ with $\varphi \in W^{s,p}$. Then for infinitely many $x'' \in \mathbb{R}^{n-2}$ and $0 < r < 1$, we have $\varphi \in W^{s,p}$ on the circle $C = \{(x_1, x_2, x'') \in \mathbb{R}^n \; ; \; x_1^2 + x_2^2 = r^2\}$. Thus, on such a C, u has a continuous phase. Equivalently, the identity map on the unit circle has a continuous phase, which is impossible.
With more work (and using degree theory for VMO-maps [17]), the above argument can be adapted to the limiting case $sp = 1$.
At the end, we find that the topological example has no lifting in $W^{s,p}$ when $1 \leq sp < 2$.

The remaining part of this paper describes results of Nguyen [22], the author [20] and of Brezis, Nguyen and the author [15], which provide the structure of $X_{s,p}$ in the region which is not covered by Theorem 1.2. Roughly speaking, the description is summarised by the following

"Theorem". The analytical and the topological example are the only obstructions to existence of lifting.

In order to give the "theorem" a rigourous meaning, it will be convenient to split the region concerned by Theorem 1.2 in three regions (see Figure 1):

$$\{(s,p) \; ; \; 0 < s < 1, 1 \leq sp < n\} \bigcup \{(s,p) \; ; \; s \geq 1, 1 \leq sp < 2\} = A \cup B \cup C,$$

where

$$A := \{(s,p)\,;\, 0 < s < 1, 2 \leq sp < n\}, B := \{(s,p)\,;\, 0 < s < 1, 1 \leq sp < 2\}, C := \{(s,p)\,;\, s \geq 1, 1 \leq sp < 2\}.$$

It will turn out that the rigourous statements depend on the region.
Note that:

(1) In region A, we have only the analytical example;
(2) In region C, we have only the topological example;
(3) In region B, the analytical and the topological example coexist.

We start by considering region A. In that region the "theorem" becomes

Theorem 1.3. [22], [20] *Assume that* $0 < s < 1$, $1 \leq p < \infty$ *and* $2 \leq sp < n$. *Then*

$$X_{s,p} = \{e^{\imath\varphi} \; ; \; \varphi \in W^{s,p} + W^{1,sp}(\mathbb{S}^n; \mathbb{R})\}.$$

The next section introduces a notion essential to the statement of the "theorem" in the regions B and C.

2. The singular set of an $\mathbb{S}^1$-valued map

Consider the maps $u, v : \mathbb{D} \to \mathbb{S}^1$, $u(z) = z/|z|$, $v(z) = e^{\imath/\sqrt{|z|}}$. (Here, $\mathbb{D}$ is the unit disk.) It is easy to check that both maps are in $W^{1,1}$.
Riddle: Which is the singular set of u, respectively v? (Answer at the end of this section.)
In order to define the right notion of singular set, we start by considering maps $u \in \mathscr{R}$, where

$$\mathscr{R} := \big\{u \in C^\infty(\mathbb{S}^n \setminus \Sigma\,; \mathbb{S}^1)\;; \Sigma = \Sigma(u) \text{ is an oriented } (n-2)-\text{submanifold of } \mathbb{S}^n, \\ \exists\ C = C(u) \text{ such that } |D^2 u(x)| \leq C/\text{dist}^2(x, \Sigma), \forall\ x \in \mathbb{S}^n\big\}.$$

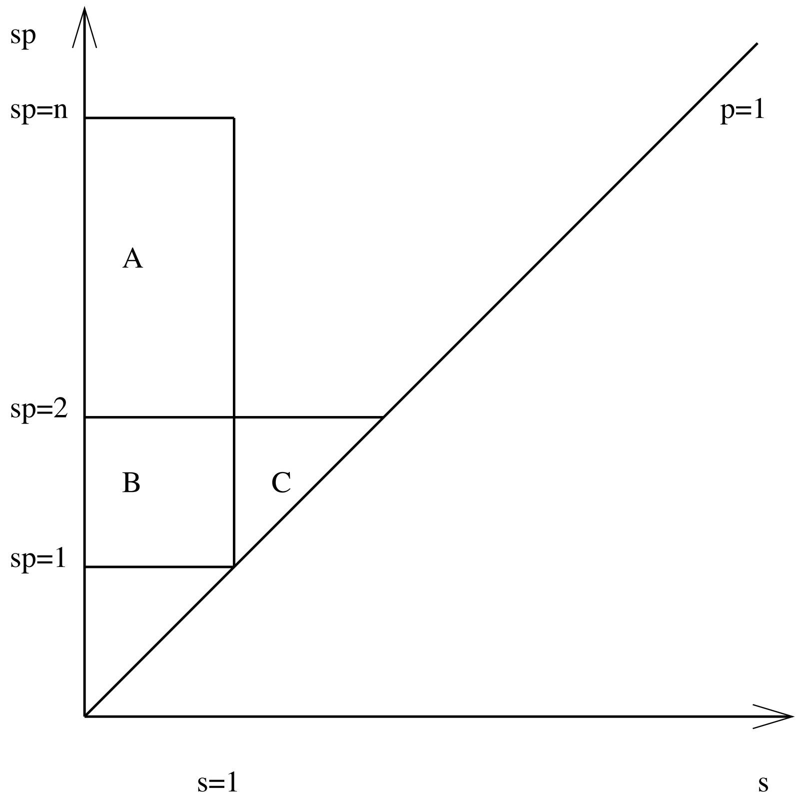

FIGURE 1. Outside $A \cup B \cup C$, $W^{s,p}$-maps do have $W^{s,p}$-phases

The following result is easily established.

Lemma 2.1. [15] *$\mathscr{R} \subset X_{s,p}$ when $sp < 2$.*

Much more delicate is

Theorem 2.1. [5], [24], [8], [10] *$\mathscr{R}$ is dense in $X_{s,p}$ whenever $1 \le sp < 2$.*

The above statement is still true when $sp < 1$. However, we have the stronger property

Theorem 2.2. [19] *$C^\infty(\mathbb{S}^n; \mathbb{S}^1)$ is dense in $X_{s,p}$ whenever $sp < 1$.*

We next take a closer look to the singular set $\Sigma = \Sigma(u)$ of a map $u \in \mathscr{R}$. We start with the case $n = 2$. In that case, Σ is a finite set, $\Sigma = \{a_1, \ldots, a_k\}$. We may associate to each a_j a degree as follows: we consider, for small $r > 0$, the restriction of u to the geodesic circle $C(a_j, r)$ of radius r around a_j. The orientation on $\mathbb{S}^2$ (say, induced by the outward normal to the unit ball) induces one on the geodesic disc $D(a_j, r)$. In turn, this induces an orientation on $C(a_j, r)$, viewed as the boundary of $D(a_j, r)$. With respect to this orientation, the smooth map $u : C(a_j, r) \to \mathbb{S}^1$ has a winding number (degree) independent of small r. We denote it by deg (u, a_j). The object that will play a crucial role in our description of $X_{s,p}$ is

$$T(u) := \sum_{j=1}^{k} \text{deg } (u, a_j)\delta_{a_j}. \tag{2.1}$$

For further use, we note that $\sum_{j=1}^{k} \text{deg } (u, a_j) = 0$.[1] Thus, possibly after relabelling the singularities of u, we may write

$$T(u) = \sum_{j=1}^{k} (\delta_{P_j} - \delta_{N_j}) \tag{2.2}$$

for some appropriate $k \in \mathbb{N}$, $P_j, N_j \in \mathbb{S}^2$.
We note that $T(u)$ does not track **all** the singularities of u: a zero degree singularity will not appear in the formula giving $T(u)$. As we will see later, non zero degree singularities are the only one relevant to analysis purposes.
We next define $T(u)$ when $n \geq 3$. In that case, if $u \in \mathscr{R}$, then $\Sigma(u)$ is a finite union of disjoint smooth oriented connected $(n-2)$-manifolds, say $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_k$. In this case, the winding number of u around each Σ_j is defined as follows: let

$$U_\varepsilon := \{y \in \mathbb{S}^n \ ; \ \text{dist } (y, \Sigma_j) \leq \varepsilon\}.$$

Here, dist stands for the geodesical distance. For small ε and for each $x \in \Sigma_j$, the set

$$D(x, r) := \{y \in \Sigma_j \ ; \ \text{dist } (y, x) \leq \varepsilon\}$$

is a 2-manifold whose boundary, $C(x, r)$, is diffeomorphic with a circle. If we consider an orientation τ_Σ on Σ_j, then we may define an orientation on $C(x, r)$ as follows: we first consider an orientation τ_D on $D(x, r)$ such that $\tau_\Sigma \wedge \tau_D$ be the natural orientation on $\mathbb{S}^n$. This induces an orientation on $C(x, r)$ (considered as the boundary of $D(x, r)$). We then let deg (u, Σ_j) be the winding number of u on $C(x, r)$ with respect to the above orientation. One may prove [10] that this definition is independent of small ε and of $x \in \Sigma_j$. We may now define

$$T(u) := \sum_{j=1}^{k} \text{deg } (u, a_j) \int_{\Sigma_j}. \tag{2.3}$$

Here, $\int_{\Sigma_j}$ denotes the integration of $(n-2)$-forms on the oriented $(n-2)$-manifold Σ_j. In other words, $T(u)$ is an $(n-2)$-current.
The key remark for defining $T(u)$ for a general u appeared in [12] in the context of $\mathbb{S}^2$-valued maps: there is a tractable formula giving $T(u)$ when $u \in \mathscr{R}$. More specifically, when $n = 2$ we have

$$\langle T(u), \zeta \rangle = \frac{1}{2\pi} \int_{\mathbb{S}^2} (u_1 du_2 - u_2 du_1) \wedge d\zeta, \qquad \zeta \in C^\infty(\mathbb{S}^2; \mathbb{R}) \equiv \Lambda^0(\mathbb{S}^2). \tag{2.4}$$

Here, d stands for the (exterior) differential and $\wedge$ is the exterior product of 1-forms (thus the integrand is a 2-form).

[1] This follows from formula (2.4) below applied to $\zeta = 1$.

For arbitrary n, the corresponding formula is

$$\langle T(u), \zeta\rangle = \frac{1}{2\pi}\int_{\mathbb{S}^n} (u_1 du_2 - u_2 du_1)\wedge d\zeta, \qquad \zeta \in \Lambda^{n-2}(\mathbb{S}^n). \tag{2.5}$$

In this case, the integrand is the exterior product of a 1-form and of an $(n-1)$-form. It follows from (2.5) that the map $u \mapsto T(u)$, initially defined for $u \in \mathscr{R}$ with values into the space $\mathcal{D}_{n-2}$ of $(n-2)$-currents, extends by density and continuity to $X_{1,1}$. A more involved result asserts that $T(u)$ may be defined beyond the space $X_{1,1}$.

Theorem 2.3. [9] *Let s, p be such that $1 \le sp < 2$. Then the map $\mathscr{R} \ni u \mapsto T(u) \in \mathcal{D}_{n-2}$ admits a unique continuous exension $T : X_{s,p} \to \mathcal{D}_{n-2}$.*
In addition, $T(u)$ does not depend on s and p, in the sense that, if $u \in X_{s_j,p_j}$, $j = 1, 2$, then $T(u)$ defined with respect to X_{s_1,p_1} equals $T(u)$ defined with respect to X_{s_2,p_2}.

We end this section by a result that illustrates the fact that the non zero degree singularities are the only one relevant.

Theorem 2.4. [18], [23], [10], [15] *Assume that $1 \le sp < 2$. Then*

$$\{u \in X_{s,p}\ ;\ T(u) = 0\} = \overline{C^\infty(\mathbb{S}^n; \mathbb{S}^1)}^{W^{s,p}}.$$

We emphasise the fact that the above result concerns density of smooth $\mathbb{S}^1$-valued maps. Each map in $X_{s,p}$ can be approximated, in the $W^{s,p}$-norm, by smooth maps. But these maps need not be $\mathbb{S}^1$-valued. Actually, one may prove[2] that $C^\infty(\mathbb{S}^n; \mathbb{S}^1)$ is never dense in $X_{s,p}$ when $1 \le sp < 2$.

Answer to the riddle: The singular set of u is 0 (with winding number 1), more specifically we have $T(u) = \delta_0$. The singular set of v is empty, in the sense that $T(v) = 0$. Consequently, the singularity of v is an illusion: v can be approximated, in $W^{1,1}$, by smooth maps. On the other hand, the singularity of u is robust: if $\{u_k\}$ is a sequence of $\mathbb{S}^1$-valued maps such that $u_k \to u$ in $W^{1,1}$, then, for large k, u_k is not smooth near the origin. Alternatively, given any neighbourhood V of the origin in $\mathbb{R}^2$, there is some $\varepsilon = \varepsilon(V)$ such that if $w \in W^{1,1}(\mathbb{D}; \mathbb{S}^1)$ and $\|Dw - Du\|_{L^1} \le \varepsilon$, then w cannot be smooth in V.

3. Singular sets for maps with prescribed regularity

In this section, we describe the image of $X_{s,p}$ under T.
The starting point is the fact that, when $u \in X_{1,1}$, $T(u)$ acts on Lipschitz forms. More specifically, it is clear from (2.5) that the map $X_{1,1} \ni u \mapsto T(u) \in (W^{1,\infty})^*$ is continuous. Here, $W^{1,\infty}$ denotes the space of Lipschitz $(n-2)$-forms. We take as semi-norm of a form $\zeta \in W^{1,\infty}$ the best Lipschitz constant of its coefficients (computed in a finite number of charts).
Let now, for $n = 2$,

$$\mathscr{E} := \left\{ \sum_{j=1}^{k} (\delta_{P_j} - \delta_{N_j})\ ;\ k \in \mathbb{N}, P_j, N_j \in \mathbb{S}^2 \right\}.$$

Formula (2.2) implies that $T(\mathscr{R}) \subset \mathscr{E}$. Using the density of $\mathscr{R}$ in $X_{1,1}$, we find that $T(X_{1,1}) \subset \overline{\mathscr{E}}^{(W^{1,\infty})^*}$. The reversed inclusion is true.

[2] By adapting the arguments in [25].

Theorem 3.1. [12], [1], [16] *The following assertions are equivalent for* $T \in \mathscr{D}'(\mathbb{S}^2) \equiv \mathscr{D}_0(\mathbb{S}^2)$*:*

(1) *There are two sequences* $(P_j), (N_j) \in \mathbb{S}^2$ *such that* $\sum_{j=1}^{\infty} |P_j - N_j| < \infty$ *and* $T = \sum_{j=1}^{\infty} (\delta_{P_j} - \delta_{N_j})$*;*

(2) *There is some* $u \in X_{1,1}$ *such that* $T(u) = T$*;*

(3) $T \in \overline{\mathscr{E}}^{(W^{1,\infty})^*}$.

In particular, $T(X_{1,1}) = \overline{\mathscr{E}}^{(W^{1,\infty})^*}$.

In higher dimensions, it is not known[3] whether the formal analog of Theorem 3.1 is true. The statement known to be true is the following

Theorem 3.2. [1] *The following assertions are equivalent for* $T \in \mathscr{D}_{n-2}(\mathbb{S}^n)$*:*

(1) *There is some rectifiable* $(n-1)$*-current* N *such that* $\partial N = T$*;*

(2) *There is some* $u \in X_{1,1}$ *such that* $T(u) = T$*.*

In (1), the boundary of N is taken in the sense of currents, i. e. ∂N is the $(n-2)$-current acting according to the formula

$$\langle \partial N, \zeta \rangle = \langle N, d\zeta \rangle, \qquad \forall \ \zeta \in \Lambda^{n-2}(\mathbb{S}^n).$$

It is known that, in two-dimensions, (1) in Theorem 3.2 is precisely (1) in Theorem 3.1.

We set, for $n \geq 2$,

$$\mathscr{E}_1 := \{\partial M \ ; \ M \text{ is a rectifiable } (n-1) - \text{current}\}$$

and, for $n \geq 3$,

$$\mathscr{E} := \left\{ \sum_{j=1}^{k} d_j \int_{\Sigma_j} \ ; \ k \in \mathbb{N}, d_j \in \mathbb{Z}, \Sigma_1, \ldots, \Sigma_k \text{ boundaryless disjoint oriented } (n-2) - \text{submanifolds of } \mathbb{S}^n \right\}.$$

It follows from Theorems 3.1-3.2 that $T(X_{1,1}) = \mathscr{E}_1$ and[4] $\mathscr{E}_1 \subset \overline{\mathscr{E}}^{(W^{1,\infty})^*}$.

We next turn to the description of $T(X_{1,p})$ when $1 < p < 2$. We note that, if $u \in X_{1,p}$, then $u_1 du_2 - u_2 du_1 \in L^p$, so that $T(u)$ acts on $W^{1,p'}$, the space of $(n-2)$-forms with coefficients in $W^{1,p'}$. (Here, p' is the conjugate of p.) If we endow $W^{1,p'}$ with the semi-norm $\zeta \mapsto \|d\zeta\|_{L^{p'}} + \|\delta\zeta\|_{L^{p'}}$[5] and repeat the argument preceding the statement of Theorem 3.1, then we find that

$$T(X_{1,p}) \subset \mathscr{E}_p := \overline{\mathscr{E}}^{(W^{1,p'})^*}.$$

The reversed inclusion is true.

Theorem 3.3. [10] *Let* $1 < p < 2$*. The following assertions are equivalent for* $n \geq 2$ *and* $T \in \mathscr{D}_{n-2}(\mathbb{S}^n)$*:*

(1) $T \in \mathscr{E}_p$*;*

(2) *There is some* $u \in X_{1,p}$ *such that* $T(u) = T$*.*

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[3] To the best of the knowledge of the author.

[4] Using the fact that $T(u) \in \mathscr{E}$ if $u \in \mathscr{R}$, and the density of $\mathscr{R}$ in $X_{1,1}$.

[5] Here, δ is the formal adjoint of the exterior differential d.

We next turn to the image of $X_{s,p}$ under T when $s \neq 1$. We start by examining the case $s > 1$. If $u \in X_{s,p}$, then $u \in W^{1,sp}$ (by Lemma 1.1), so that $u_1 du_2 - u_2 du_1$ acts on $W^{1,(sp)'}$-forms. On the other hand, a standard result on the regularity of products yields [7]

$$u \in W^{s,p} \cap L^\infty,\ s > 1 \Longrightarrow u_1 du_2 - u_2 du_1 \in W^{s-1,p},$$

so that $T(u)$ acts on forms ζ such that $d\zeta \in W^{1-s,p'}$. In particular, $T(u)$ acts on $W^{2-s,p'}$. Using the density of $\mathscr{R}$ in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$, we find that, when $s > 1$ and $1 < sp < 2$, we have

$$T(X_{s,p}) \subset \overline{\mathscr{E}}^{(W^{1,(sp)'})^* \cap (W^{2-s,p'})^*}.$$

The reversed inclusion holds also

Theorem 3.4. [10] *We have* $T(X_{s,p}) = \overline{\mathscr{E}}^{(W^{1,(sp)'})^* \cap (W^{2-s,p'})^*}$.

Our next result is, in some sense, function theoretic, and has no *a priori* connection to $\mathbb{S}^1$-valued maps. However, its only known proof relies heavily on tools tailored specifically for $\mathbb{S}^1$-valued maps.

Theorem 3.5. [15] *Let* $s > 1$, $1 < sp < 2$. *Then we have*

$$\overline{\mathscr{E}}^{(W^{1,(sp)'})^* \cap (W^{2-s,p'})^*} = \overline{\mathscr{E}}^{(W^{1,(sp)'})^*} = \mathscr{E}_{sp}.$$

More specifically, we have, for $T \in \mathscr{E}$, *the estimate*

$$\|T\|_{(W^{2-s,p'})^*} \le C \|T\|^s_{(W^{1,(sp)'})^*}. \tag{3.1}$$

Consequently, the previous theorem can be improved to $T(X_{s,p}) = \mathscr{E}_{sp}$.

There is something intriguing about estimate (3.1). One may see, using Sobolev non embeddings, that there is no inclusion relation between $W^{1,(sp)'}$ and $W^{2-s,p'}$; similarly, there is no inclusion relation between there duals $(W^{1,(sp)'})^*$ and $(W^{2-s,p'})^*$. Estimate (3.1) does not hold for arbitrary functionals T, for otherwise we would have $(W^{1,(sp)'})^* \subset (W^{2-s,p'})^*$. It has already been noted (in the context of the Ginzburg-Landau equation [4]) that distributions with "small" support satisfy improved estimates (see also [21] for an abstract approach). The situation we encounter here seems to be somewhat similar, since when $T \in \mathscr{E}$, T is supported in an $(n-2)$-manifold.[6]

Open Problem 1. In 2D, (3.1) is equivalent to the following estimate: let $P_j, N_j \in \mathbb{S}^2$, $j = 1, \ldots, m$. Let $1 < q < 2$. Then

$$\left\| \sum_{j=1}^m (\delta_{P_j} - \delta_{N_j}) \right\|_{(C^{0,2-q})^*} \le C_q \left\| \sum_{j=1}^m (\delta_{P_j} - \delta_{N_j}) \right\|^q_{(W^{1,q'})^*}. \tag{3.2}$$ [7]

Question: is there any short proof of (3.2)?
Similar question in 3+D, when sums of Dirac masses are replaced by $T \in \mathscr{E}$ and the spaces $C^{0,2-q}$ and $W^{1,q'}$ are spaces of $(n-2)$-forms.

[6] This is exactly the dimension of singular sets observed in the study of the Ginzburg-Landau equation.

[7] This is a fact, not an open problem.

Open Problem 2. The formal generalization of (3.2) for higher codimensional currents is the following: let $k \geq 2$. Let $P_j, N_j \in \mathbb{S}^{k+1}$, $j = 1, \ldots, m$. Let $k < q < k+1$. Then

$$\left\| \sum_{j=1}^{m} (\delta_{P_j} - \delta_{N_j}) \right\|_{(C^{0,k+1-q})^*} \leq C_q \left\| \sum_{j=1}^{m} (\delta_{P_j} - \delta_{N_j}) \right\|_{(W^{1,(q/k)'})^*}^{q/k}. \tag{3.3}$$

Is (3.3) true?

Same question when $\mathbb{S}^{k+1}$ is replaced by $\mathbb{S}^n$ with $n \geq k+2$ and $T = \displaystyle\sum_{j=1}^{m} (\delta_{P_j} - \delta_{N_j})$ is replaced by $T = \displaystyle\sum_{j=1}^{m} d_j \int_{\Sigma_j}$, with $d_j \in \mathbb{Z}$ and the Σ_j's are $(n-k-1)$-submanifolds of $\mathbb{S}^n$.

Open Problem 3. Is there an "abstract" form of (3.2) and (3.3) in the spirit of [21]?

We now turn to the description of $T(X_{s,p})$ when $s < 1$. Assume first that $1 < sp < 2$. Then $X_{1,sp} \subset X_{s,p}$, by Lemma 1.1. By Theorem 3.1, it follows that $T(X_{s,p}) \supset \mathscr{E}_{sp}$. The reversed inclusion is true.

Theorem 3.6. [15] *Assume that* $0 < s < 1$ *and* $1 < sp < 2$. *Then* $T(X_{s,p}) = \mathscr{E}_{sp}$.

When $sp = 1$, there is no obvious relation between $T(X_{s,p})$ and $\mathscr{E}_1$. However, we have

Theorem 3.7. [11] *Assume that* $0 < s < 1$ *and* $sp = 1$. *Then* $T(X_{s,p}) = \mathscr{E}_1$.

Theorems 3.1, 3.2, 3.3, 3.5, 3.6, 3.7 yield the following short conclusion

Theorem 3.8. *Assume that* $1 \leq sp < 2$. *Then* $T(X_{s,p}) = \mathscr{E}_{sp}$.

4. Singular sets for maps with prescribed regularity: how the proofs work

In this section, we briefly sketch the arguments leading to Theorem 3.8.

The key ingredient is the following

Theorem 4.1. [20], [15] *Assume that* $1 \leq sp < 2$ *and let* $u \in X_{s,p}$. *Then there is some* $\varphi \in W^{s,p}$ *such that* $u_1 du_2 - u_2 du_1 - d\varphi \in W^{sp-1,1}$.
Equivalently: there is some $\varphi \in W^{s,p}$ *such that the map* $v := ue^{-\imath\varphi}$ *is in* $W^{sp,1}$.

We note that the object $u_1 du_2 - u_2 du_1 - d\varphi$ is well-defined (as a current) in view of Theorem 2.3. The proof of the above theorem is constructive[8] and inspired by [6], which contains a variant of Theorem 4.1 in the special case $s = 1/2$, $p = 2$.

Sketch of proof of Theorem 3.8 (assuming Theorem 4.1). Assume that $1 \leq sp < 2$. Let $u \in W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ and let φ be as in Theorem 4.1. Recall that[9] we have

[8] There is an explicit formula for φ in terms of u.
[9] When u is sufficiently smooth, say at least $W^{1,1}$.

$\langle T(u),\zeta\rangle = \dfrac{1}{2\pi}\displaystyle\int_{\mathbb{S}^n}(u_1du_2-u_2du_1)\wedge d\zeta$. On the other hand, it is easy to justify the equality $v_1dv_2-v_2dv_1=u_1du_2-u_2du_1-d\varphi$. Thus[10]

$$\begin{aligned}\langle T(ue^{-\imath\varphi}),\zeta\rangle &= \frac{1}{2\pi}\int_{\mathbb{S}^n}(u_1d_2-u_2du_1-d\varphi)\wedge d\zeta\\ &= \langle T(u),\zeta\rangle - \frac{1}{2\pi}\int_{\mathbb{S}^n}d\varphi\wedge d\zeta = \langle T(u),\zeta\rangle.\end{aligned}\tag{4.1}$$

In other words, we have $T(u)=T(v)$. Since, by Theorem 4.1, we have $v\in W^{sp,1}$, we find that

$$T(X_{s,p})\subset T(X_{sp,1}).\tag{4.2}$$

When $1<sp<2$, Lemma 1.1 yields $X_{sp,1}\subset X_{s,p}$, so that $T(X_{s,p})\supset T(X_{sp,1})$. Using the trivial remark $T(X_{1,q})\subset\mathscr{E}_q$, $1<q<2$, we find that

$$T(X_{s,p})=T(X_{sp,1})=T(X_{1,sp})\subset\mathscr{E}_{sp}.\tag{4.3}$$

The reversed inclusion $T(X_{1,sp})\subset\mathscr{E}_{sp}$ is Theorem 3.3. We will return to its proof few lines later.

The case $sp=1$ needs a different argument. In that case, (4.2) still holds. However, since $X_{1,1}\not\subset X_{s,1/s}$ when $0<s<1$, the reversed inclusion $T(X_{1,1})\subset T(X_{s,1/s})$ is not obvious. This inclusion is true and obtained *via* a generalized "dipole construction" (in the spirit of [12]; see also [2], [1]). This construction yields the "converse" to Theorem 4.1 in the limiting case $sp=1$: given $v\in X_{1,1}$, there is some $\varphi\in W^{1,1}+W^{s,p}$ such that $u:=ve^{\imath\varphi}\in X_{s,p}$. Arguing as in (4.1), we find that $T(X_{1,1})\subset T(X_{s,1/s})$.

The final ingredient in the proof of Theorem 3.8 is the inclusion $T(X_{1,q})\supset\mathscr{E}_q$, $1\le q<2$ (Theorems 3.2 and 3.3). For $q=1$, this is obtained through a dipole construction [1]. In the case $1<q<2$, the construction is different [10] and relies on elliptic equations and elliptic regularity. Given $T\in\mathscr{E}_q$, we let $F\in\mathscr{D}_{n-2}$ solve $\Delta F=2\pi T$, where Δ is the Laplace-Beltrami operator on $\mathbb{S}^n$. By standard elliptic estimates, F belongs to $W^{1,q}$. The key step is to prove existence of some $\mathbb{S}^1$-valued u such that $\dfrac{1}{\imath u}du=u_1du_2-u_2du_1=(-1)^n * dF$. Here, $*$ is the Hodge operator, so that $*dF$ is an 1-form. Since[11] $|du|=|dF|$, we find that $u\in W^{1,q}$. A straightforward calculation shows that $T(u)=T$. □

Remark 4.1. We take here a closer look to the proof of Theorem 3.3, i. e., to the construction $T\mapsto F\mapsto u$ described right above. The following two dimensional example will give better insight to this construction. It will be more convenient to work on $\mathbb{R}^2$ rather than $\mathbb{S}^2$. In order to construct a map u such that $T(u)=T:=\delta_0$, we proceed as follows:

(1) We solve $\Delta F=2\pi\delta_0$. "The" solution is $F=\ln|x|$.
(2) We solve $u_1du_2-u_2du_1=dF^\perp$. Here, $^\perp$ stands for the counterclockwise rotation of $\dfrac{\pi}{2}$. If u is smooth, then we may write $u=e^{\imath\varphi}$ with smooth φ and equation $u_1du_2-u_2du_1=dF^\perp$ becomes $d\varphi=dF^\perp$. In our case, u

[10] Formally. However, one may rigourously prove that the conclusion of (4.1) is correct.
[11] At least formally.

can not be smooth, but we may see that "the" solution is $\varphi = \theta$ (the polar angle). This gives $u(z) = e^{\imath\varphi} = e^{\imath\theta} = \dfrac{z}{|z|}$.

(3) We have, for this u, $T(u) = T$.

Noting that θ and ln are harmonic conjugated, we see that the proof of Theorem 3.3 consists roughly speaking in finding the harmonic conjugate of the phase of u. The advantage of this approach is that it allows to work with a global object (ln) instead of a local one (θ).

Thus the argument giving $T(X_{1,q})$, $1 < q < 2$ (Theorem 3.3), uses elliptic estimates, while the one that yields $T(X_{1,1})$ (Theorem 3.2) is based on a direct construction. There is some hope to unify the two proofs. This would require a new elliptic estimate that eludes us.

Open Problem 4. Let M be a rectifiable $(n-1)$-current on $\mathbb{S}^n$. Let $T = \partial M \in \mathscr{D}_{n-2}(\mathbb{S}^n)$. Let $F \in \mathscr{D}_{n-2}(\mathbb{S}^n)$ solve $\Delta F = T$. Is it true that $\|dF\|_{L^1} \leq C|M|$? More generally, let $1 < p < \infty$. Is it true that $|F|^p_{W^{1/p,p}} \leq C_p|M|$?

Here, $|M|$ stands the mass of M.[12] One could ask the same question for currents of arbitrary dimension.

Remark 4.2. For a better understanding of the "Theorem", we will give here a (not so) naive interpretation of Theorem 3.3.
We start with the dimension 2 and replace, for simplicity, $\mathbb{S}^2$ by the unit disc $\mathbb{D}$. By (the analog in $\mathbb{D}$ of) Theorem 3.1, each $T \in \mathscr{E}_1$ (and thus each $T \in \mathscr{E}_p$, where $1 < p < 2$) is of the form $T = \displaystyle\sum_{j=1}^{\infty}(\delta_{P_j} - \delta_{N_j})$, where $P_j, N_j \in \overline{\mathbb{D}}$ satisfy $\displaystyle\sum_{j=1}^{\infty}|P_j - N_j| < \infty$.

Assume first that $T \in \mathscr{E}_p$ is given by a finite sum: $T = \displaystyle\sum_{j=1}^{k}(\delta_{P_j} - \delta_{N_j})$. Adapting to the case of $\mathbb{D}$ the proof of Theorem 3.3, we may see, in the spirit of Remark 4.1, that "the" map $u \in W^{1,p}$ such that $T(u) = T$ is of the form $u = \displaystyle\prod_{j=1}^{k}\left(\frac{z-P_j}{|z-P_j|}\frac{|z-N_j|}{z-N_j}\right)e^{\imath\psi}$, where ψ is smooth.

If we allow T to be a sum with infinitely many terms, then the product $\displaystyle\prod_{j=1}^{\infty}\left(\frac{z-P_j}{|z-P_j|}\frac{|z-N_j|}{z-N_j}\right)$ need not converge. Instead, one may write T as $T = \displaystyle\prod_{j=1}^{\infty}\left(\frac{z-P_j}{|z-P_j|}\frac{|z-N_j|}{z-N_j}e^{\imath\psi_j}\right)$, where the smooth phases ψ_j make the product converge. Thus one may see u as a

[12] In the special case where M is then integration over a smooth oriented $(n-1)$-manifold Σ, $|M|$ is the $(n-1)$-dimensional Hausdorff measure of Σ. In general, $|M|$ is the total variation of the measure M.

corrected infinite product of terms of the form $\left(\dfrac{z-a}{|z-a|}\right)^{\pm 1}$.[13]
In dimension $n \geq 3$, points are replaced by smooth oriented $(n-2)$-manifolds, say $\Sigma_j,\ j \geq 1$. In this case, one could interpret u as a corrected infinite product of terms that, near a point $x_0 \in \Sigma_j$, are of the form $\dfrac{(\Phi_1(x), \Phi_2(x))}{|(\Phi_1(x), \Phi_2(x))|}$.
Here, Φ is a local diffeomorphism from a neighborhood of x_0 in $\mathbb{R}^n$ to a neighborhood of the origin in $\mathbb{R}^n$ and Φ flattens, locally around x_0, Σ_j to $\{(0,0)\} \times \mathbb{R}^{n-2}$.

The proofs of Theorems 3.8 and 3.3 have the following pleasant byproduct.

Theorem 4.2. *Assume that* $1 < q < 2$. *Then there is a map*

$$\mathscr{E}_q \ni T \overset{\Phi}{\mapsto} u \in W^{q,1}(\mathbb{S}^n; \mathbb{S}^1)/\mathbb{S}^1$$

such that

1. $T(\Phi(T)) = T,\ \forall\ T \in \mathscr{E}_q$;
2. Φ *is continuous;*
3. $\Phi(T+S) = \Phi(T)\Phi(S),\ \forall\ S, T \in \mathscr{E}_q$.

Here, $W^{q,1}(\mathbb{S}^n; \mathbb{S}^1)/\mathbb{S}^1$ is $W^{q,1}(\mathbb{S}^n; \mathbb{S}^1)$ modulo constants.

Sketch of proof. If $F = F(T)$ is "the"[14] $(n-2)$-current that solves $\Delta F = T$, then the map $T \mapsto F$ is linear. The map $u = \Phi(T)$ solves $u_1 du_2 - u_2 du_1 = F$. It is easy to see that, up to a multiplicative constant, this equation has at most one solution.[15] This implies that $\Phi(T+S) = \Phi(T)\Phi(S)$. Continuity of Φ follows from the continuity of $T \mapsto F$, which is part of the proof of Theorem 3.8. □

The Gagliardo-Nirenberg inequalities imply that, given $1 < q < 2$, the map Φ constructed in Theorem 4.2 is continuous from $\mathscr{E}_q$ into $W^{s,p}(\mathbb{S}^n; \mathbb{S}^1)$ for each s, p such that $sp = q$. In addition, $\Phi(T)$ does not depend on q, but only on T. A variant of Open Problem 4 which would allow the extension of this result to $q = 1$ is

Open Problem 5. Is there a map

$$\mathscr{E}_1 \ni T \overset{\Phi}{\mapsto} u \in \bigcap_{1 \leq p < \infty} W^{1/p,p}(\mathbb{S}^n; \mathbb{S}^1)/\mathbb{S}^1$$

such that

1. $T(\Phi(T)) = T,\ \forall\ T \in \mathscr{E}_1$;
2. Φ is continuous from $\mathscr{E}_1$ into $W^{1/p,p}(\mathbb{S}^n; \mathbb{S}^1)/\mathbb{S}^1$, $\forall\ 1 \leq p < \infty$;
3. $\Phi(T+S) = \Phi(T)\Phi(S)$, $\forall\ S, T \in \mathscr{E}_1$?

Here, we consider on $\mathscr{E}_1$ the distance

$$d(T_1, T_2) = \inf\{|M|\ ;\ \partial M = T_2 - T_1\}.$$

In connection to Open Problem 5, what is known to be true is the following very partial result

[13] The need to correct the factors is similar to the one encountered in the construction of a holomorphic function f with a given sequence of zeroes $\{z_n\}$. The answer is not $f(z) = (z-z_1)(z-z_2)\ldots$; each factor $z - z_j$ has to be corrected in order to make the product converge.

[14] When $n = 2$, uniqueness requires a normalization condition, e. g. $\int_{\mathbb{S}^2} F = 0$.

[15] In $W^{1,1}$.

Theorem 4.3. [11] *There is a map* $\Phi : \mathscr{E}_1 \to \bigcap_{1\le p<\infty} W^{1/p,p}(\mathbb{S}^n;\mathbb{S}^1)/\mathbb{S}^1$ *such that* $T(\Phi(T)) = T$ *for each* $T \in \mathscr{E}_1$.

5. Lifting for maps without singularities

We start this section by reducing the problem of the description of arbitrary maps in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ to the one of the description of maps in the closure of $C^\infty(\mathbb{S}^n;\mathbb{S}^1)$ for the $W^{s,p}$-norm.

Assume first that (s,p) is in the region A described in Section 1. Then maps in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ have no essential singularities.

Theorem 5.1. [14] *Let* $0 < s < 1$, $1 \le p < \infty$ *be such that* $2 \le sp < n$. *Then* $C^\infty(\mathbb{S}^n;\mathbb{S}^1)$ *is dense in* $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$.

We next turn to the regions B and C. The following result is easily proved.

Lemma 5.1. *Let* $1 \le sp < 2$ *and let* $u, v \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$. *Then* $T(uv) = T(u) + T(v)$ *and* $T(u\overline{v}) = T(u) - T(v)$. [16]

Let now, for $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$, $v \in W^{sp,1}(\mathbb{S}^n;\mathbb{S}^1) \cap W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ be such that $T(v) = T(u)$.[17] Lemma 5.1 and Theorem 2.4 imply that $w := u/v \in \overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$.

Thus describing maps in $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ when (s,p) is in either of the regions A or B or C amounts to describing a map in $\overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$. This is achieved by the following result, whose first part is Theorem 1.3.

Theorem 5.2. [18], [7], [11], [6], [10], [22], [20] *Let* $u \in \overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$.

(1) *Assume that* (s,p) *is in the region* A, *i. e.,* $0 < s < 1$, $2 \le sp < n$. *Then one may write* $u = e^{\imath\varphi}$ *for some real* $\varphi \in W^{s,p} + W^{1,sp}$.
The converse is true, i. e., if $\varphi \in W^{s,p} + W^{1,sp}$, *then* $u := e^{\imath\varphi}$ *belongs to* $\overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$.
(2) *Assume that* (s,p) *is in the region* C, *i. e.,* $s \ge 1$, $1 \le sp < 2$. *Then one may write* $u = e^{\imath\varphi}$ *for some real* $\varphi \in W^{s,p} \cap W^{1,sp}$.
The converse is true, i. e., if $\varphi \in W^{s,p} \cap W^{1,sp}$, *then* $u := e^{\imath\varphi}$ *belongs to* $\overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$.
(3) *Assume that* (s,p) *is in the region* B, *i. e.,* $0 < s < 1$, $1 \le sp < 2$. *Then one may write* $u = e^{\imath\varphi}$ *for some real* $\varphi \in W^{s,p} + W^{1,sp}$.
When $1 < sp < 2$, *the converse is true, i. e., if* $\varphi \in W^{s,p} + W^{1,sp}$, *then* $u := e^{\imath\varphi}$ *belongs to* $\overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$.
When $sp = 1$, *the converse is wrong, i. e. there is some* $\varphi \in W^{s,p} + W^{1,1}$ *such that* $u := e^{\imath\varphi}$ *does not belong to* $\overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{s,p}}$.[18]

Sketch of the proof. Regularity of u from the one of φ is a consequence of regularity results for superposition operators [13] and Gagliardo-Nirenberg inequalities. More delicate is existence of φ given u. Proof of existence is rather simple when

[16] This is clear when $u, v \in \mathscr{R}$. The general case follows by density+continuity.
[17] Such v exists, by Theorems 3.8 and 4.3.
[18] Actually, such u need not even belong to $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$.

$(s,p) \in C$. Assume that φ does exist. Since maps we consider are at least $W^{1,1}$, we have

$$u = e^{\imath\varphi} \Longrightarrow du = \imath e^{\imath\varphi} d\varphi \Longrightarrow d\varphi = u_1 du_2 - u_2 du_1 := X.$$

The idea is to solve the equation $d\varphi = X$. Using the fact that $u \in \overline{C^\infty(\mathbb{S}^n; \mathbb{S}^1)}^{W^{s,p}}$, one may prove that the vector field X is closed. A Poincaré type lemma for distributions implies existence of φ. Regularity of φ follows from standard regularity results on products.

More delicate are the cases where (s,p) is in A or B. In these cases, we start with a smooth map $u \in C^\infty(\mathbb{S}^n; \mathbb{S}^1)$. For such a map, we explicitely split its phase into a $W^{s,p}$-part and a $W^{1,sp}$-part, each one satisfying appropriate estimates. By a rather straightforward limiting procedure, these estimates transfer to a general map $u \in \overline{C^\infty(\mathbb{S}^n; \mathbb{S}^1)}^{W^{s,p}}$. The first phase decomposition of this kind appears in [6] and concerns $H^{1/2}$-maps. The general decomposition in [20] is inspired by the one in [6]. For simplicity, we present it for maps defined in $\mathbb{R}^n$ rather than $\mathbb{S}^n$. Assume that $u : \mathbb{R}^n \to \mathbb{S}^1$ is constant at infinity[19], e. g. we assume that $u - 1 \in W^{s,p}$. We let v be any reasonable extension of u to $\mathbb{R}^n \times [0,\infty)$. Typically, we could let $v(x,t) = u * \rho_t(x)$, where ρ is a nice mollifier. Assume first that $u - 1$ is small in absolute value. Then v is close to 1, so that we can project v onto $\mathbb{S}^1$ and obtain a map as smooth as v. We let $w := v/|v| : \mathbb{R}^n \times [0,\infty) \to \mathbb{S}^1$ be this projection. We may then write $w = e^{\imath\psi}$ with smooth ψ. Since w is close to 1 at infinity, we may pick ψ close to 1 at infinity. Then $\varphi(x) := \psi(x,0)$ is a phase of u and, assuming that the integrals in the calculation below converge, we have

$$\begin{aligned} \varphi(x) &= -\,\psi(x,t)\Big|_{t=0}^{t=\infty} = -\int_0^\infty \frac{\partial\psi}{\partial t}(x,t)\,dt \\ &= -\int_0^\infty \left(w_1 \frac{\partial w_2}{\partial t} - w_2 \frac{\partial w_1}{\partial t} \right)(x,t)\,dt. \end{aligned} \tag{5.1}$$

If $u - 1$ is not assumed to be small anymore, we can still consider the last integral in (5.1), but it need not give φ anymore.

The splitting of the phase φ of an arbitrary smooth map u is inspired by the above remark. More specifically, we let u and v as above. We define w as a sort of projection of v onto $\mathbb{S}^1$: we let $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map such that $\Pi(z) = \dfrac{z}{|z|}$ when $|z| \geq \dfrac{1}{2}$ and we set $w := \Pi(v)$. This w coincides with the previous one when u is close to 1. We set

$$\varphi_1(x) := -\int_0^\infty \left(w_1 \frac{\partial w_2}{\partial t} - w_2 \frac{\partial w_1}{\partial t} \right)(x,t)\,dt, \quad \varphi_2 := \varphi - \varphi_1. \tag{5.2}$$

With some work, one may estimate φ_1 in $W^{s,p}$ and φ_2 in $W^{1,sp}$ provided that $0 < s < 1$.[20] More specifically, we have

$$|\varphi_1|_{W^{s,p}} \leq C|u|_{W^{s,p}}, \quad \|d\varphi_2\|_{L^{sp}}^{sp} \leq C|u|_{W^{s,p}}^p \tag{5.3}$$

provided $0 < s < 1$. These estimates extend to maps u in $\overline{C^\infty(\mathbb{S}^n; \mathbb{S}^1)}^{W^{s,p}}$. □

[19] This assumption is the natural substitute for "u is compactly supported".

[20] This covers the case $(s,p) \in A \cup B$.

6. Description of $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$

We may now put together the puzzle.

Recall that, for $1<q<2$, we set[21]

$$\mathscr{E}_q := \overline{\left\{ \sum_{j=1}^k d_j \int_{\Sigma_j} \ ;\ k\in\mathbb{N}, d_j\in\mathbb{Z}, \Sigma_1,\ldots,\Sigma_k \text{ boundaryless oriented } (n-2)-\text{submanifolds of } \mathbb{S}^n \right\}}^{(W^{1,q'})^*},$$

while for $q=1$ we defined

$$\mathscr{E}_1 := \{\partial M \ ;\ M \text{ is a rectifiable } (n-1)-\text{current}\}.$$

Recall also the existence of a map $\Phi : \mathscr{E}_q \to \bigcap_{1\le p<\infty} W^{q/p,p}(\mathbb{S}^n;\mathbb{S}^1)/\mathbb{S}^1$ such that $T(\Phi(T))=T$, $\forall\, T\in\mathscr{E}_q$, where T is the map "$u\mapsto \operatorname{Sing} u$" rigourously defined *via* Theorem 2.3.

Using these objects, we reach the following description of $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$.[22]

Theorem 6.1. *Assume that $n\ge 2$, $s>0$, $1\le p<\infty$ and set*

$$X_{s,p} := \{u:\mathbb{S}^n\to\mathbb{S}^1 \ ;\ u\in W^{s,p}\}.$$

(1) *When $sp<1$ or $sp\ge n$, we have*

$$X_{s,p} = \{e^{\imath\varphi} \ ;\ \varphi\in W^{s,p}\}.$$

(2) *When $s\ge 1$ and $2\le sp<n$, we have*

$$X_{s,p} = \{e^{\imath\varphi} \ ;\ \varphi\in W^{s,p}\cap W^{1,sp}\}.$$

(3) *When $n\ge 3$, $0<s<1$ and $2\le sp<n$, we have*

$$X_{s,p} = \{e^{\imath\varphi} \ ;\ \varphi\in W^{s,p}+W^{1,sp}\}.$$

(4) *When $s\ge 1$ and $1\le sp<2$, we have*

$$X_{s,p} \approx \mathscr{E}_{sp}\times (W^{s,p}\cap W^{1,sp}),$$

in the following sense:

- *For $u\in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$, let $v:=\Phi(T(u))$.*[23] *Then u/v writes as $e^{\imath\varphi}$ for some $\varphi\in W^{s,p}\cap W^{1,sp}$. Thus, we may associate to each $u\in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ the couple $(T(u),\varphi)\in\mathscr{E}_{sp}\times(W^{s,p}\cap W^{1,sp})$.*
- *Conversely, let $(T,\varphi)\in\mathscr{E}_{sp}\times(W^{s,p}\cap W^{1,sp})$. Then $u:=\Phi(T)e^{\imath\varphi}\in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$.*[24]

(5) *When $0<s<1$ and $1<sp<2$, we have*

$$X_{s,p} \approx \mathscr{E}_{sp}\times (W^{s,p}+W^{1,sp}),$$

in the following sense:

[21] When $n=2$ and $1<q<2$, the condition $\sum_{j=1}^k d_j=0$ has to be added to the definition of $\mathscr{E}_q$.

[22] Theorem 6.1 is nothing else but the sum of the results in the previous sections.

[23] This slightly lacks of rigour, since $\Phi(T(u))$ is a class of maps. Rigourously, one should consider u/v and φ as classes of maps modulo constants. Alternatively, one may pick a representative $v\in\Phi(T(u))$ and work with the genuine maps u/v and φ.

[24] Here, it is convenient to replace $\Phi(T)$ by one of its representatives. An even more convenient way to write formulae consists in working in the quotient spaces $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)/\mathbb{S}^1$ and $(W^{s,p}\cap W^{1,sp})/\mathbb{R}$.

- *For* $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$*, let* $v := \Phi(T(u))$*. Then* u/v *writes as* $e^{\imath\varphi}$ *for some* $\varphi \in W^{s,p} + W^{1,sp}$*. Thus, we may associate to each* $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ *the couple* $(T(u),\varphi) \in \mathscr{E}_{sp} \times (W^{s,p} + W^{1,sp})$*.*
- *Conversely, let* $(T,\varphi) \in \mathscr{E}_{sp} \times (W^{s,p} + W^{1,sp})$*. Then* $u := \Phi(T)e^{\imath\varphi} \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$*.*

(6) *When* $0 < s < 1$ *and* $sp = 1$*, we may associate to each* $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ *a couple* $(T,\varphi) \in \mathscr{E}_1 \times (W^{s,p} + W^{1,1})$*, in the following sense: for* $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$*, let* $v := \Phi(T(u))$*. Then* u/v *writes as* $e^{\imath\varphi}$ *for some* $\varphi \in W^{s,p} + W^{1,1}$*.*

(7) *The converse to item (6) is wrong, i. e., if* $(T,\varphi) \in \mathscr{E}_1 \times (W^{s,p} + W^{1,1})$*, then* $u := \Phi(T)e^{\imath\varphi}$ *need not belong to* $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$*.*

(8) *There is an "exact" description of* $W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ *when* $0 < s < 1$*,* $sp = 1$*, which writes as follows: set, for* $1 < p < \infty$*,*

$$Y^p := W^{1/p,p} + \{\varphi \in W^{1,1}\ ;\ e^{\imath\varphi} \in W^{1/p,p}\}.$$

Then

$$X_{s,p} \approx \mathscr{E}_1 \times Y^p,$$

in the following sense:

- *For* $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$*, let* $v := \Phi(T(u))$*. Then* u/v *writes as* $e^{\imath\varphi}$ *for some* $\varphi \in Y^p$*. Thus, we may associate to each* $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ *the couple* $(T(u),\varphi) \in \mathscr{E}_1 \times Y^p$*.*
- *Conversely, let* $(T,\varphi) \in \mathscr{E}_1 \times Y^p$*. Then* $u := \Phi(T)e^{\imath\varphi} \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$*.*

The drawback of item (8) is that the there is no explicit description of Y^p. Only some straightforward properties of Y^p are known.

Lemma 6.1. [15] *For* $1 < p < \infty$ *we have:*

(1) $Y^p = \{\varphi \in W^{1/p,p} + W^{1,1}\ ;\ e^{\imath\varphi} \in \overline{C^\infty(\mathbb{S}^n;\mathbb{S}^1)}^{W^{1/p,p}}\}$*;*
(2) $\varphi,\psi \in Y^p \Longrightarrow \varphi \pm \psi \in Y^p$*.*

We will return later to the interest of the following

Open Problem 6. (1) Is there a description of Y^p in terms of "usual" function spaces?
(2) Is it true that Y^p is a vector space?

Remark 6.1. The statement of Theorem 6.1, especially items (4) and (5), is indeed the rigourous form of the "Theorem". To see this, consider, e. g., the item (4). For simplicity, we let $n = 2$ and replace $\mathbb{S}^2$ by $\mathbb{D}$. By Remark 4.2, the map v in (4) is a corrected infinite product of maps of the form $\left(\dfrac{z-a}{|z-a|}\right)^{\pm 1}$. On the other hand, the map w in (4) is of the form $w = fg$, where $f := e^{\imath\psi}$, $g := e^{\imath\varphi}$, with $\psi \in W^{1,sp}$ and $\varphi \in W^{s,p}$. Thus we may write each $u \in W^{s,p}(\mathbb{S}^n;\mathbb{S}^1)$ as $u = v\, f\, g$, where v is (a corrected infinite) product of terms of the form $\left(\dfrac{z-a}{|z-a|}\right)^{\pm 1}$, f has a $W^{1,sp}$-phase and g does lift in $W^{s,p}$.
Similar considerations apply to the item (5).

7. An application: the square root problem

In [3], the authors addressed (and partially solved) the following problem: given $n \geq 2$, $s > 0$ and $1 \leq p < \infty$, decide whether each map $u \in X_{s,p}$ has a square root $v \in X_{s,p}$. This problem is completely solved (*via* Theorem 6.1) in [15]. Following [15], we discuss here a sharper version of this question, which is not completely solved:

$(Q_{s,p,u})$ Given $n \geq 2, s > 0, 1 \leq p < \infty$ and $u \in X_{s,p},$ is there some $v \in X_{s,p}$ such that $v^2 = u$?

In order to introduce our result, let us consider some examples. Let first $u_1 : \mathbb{D} \to \mathbb{S}^1$, $u_1(z) = \dfrac{z}{|z|}$. It is easy to see that $u_1 \in W^{1,1}$. However, u_1 has no square root $v \in W^{1,1}$. Indeed, argue by contradiction: if such v exists then, then on a "generic" circle $C(0,r)$ we would have $v \in C^0(C(0,r))$, so that, for such r, it holds[25]

$$1 = \text{deg }(u_1, C(0,r)) = \text{deg }(v^2, C(0,r)) = 2\text{deg }(v, C(0,r)),$$

a contradiction.

Similarly, a map with a singular point of odd winding number has no square root in $W^{1,1}$. On the other hand, the map $u_2 : \mathbb{D} \to \mathbb{S}^1$, $u_2(z) = \dfrac{z^2}{|z|^2}$, whose singularity has an even multiplicity, has the obvious square root $z \mapsto \dfrac{z}{|z|}$.

The difference between these two examples is that $T(u_1) = \delta_0$, which is an "odd multiplicity" singular set, while $T(u) = 2\delta_0$, which is an "even multiplicity" singular set.

The partial answer to $Q_{s,p,u}$ is given by the following

Theorem 7.1. (1) *When $sp < 1$ or $sp \geq 2$, the answer to $Q_{s,p,u}$ is yes.*
(2) *When $1 < sp < 2$ or $s = p = 1$, the answer to $Q_{s,p,u}$ is yes if and only if "the singular set of u is even", i. e., if and only if $T(u) \in 2\mathscr{E}_{sp}$.*
(3) *When $0 < s < 1$ and $sp = 1$, the answer to $Q_{s,p,u}$ is no if "the singular set of u is odd", i. e., if $T(u) \notin 2\mathscr{E}_{sp}$.*

The missing result is the converse to item (3); if this converse were true, this would imply that item (2) still holds in the limiting case $sp = 1$.

Proof. When $sp < 1$ or $sp \geq 2$, the conclusion is immediate, *via* Theorem 6.1.
We assume that $1 \leq sp < 2$. By Lemma 5.1, we have $T(uv) = T(u) + T(v)$, $\forall\ u, v \in X_{s,p}$. This implies at once that, if the answer to $Q_{s,p,u}$ is yes, then $T(u) \in 2\mathscr{E}_{sp}$.
Conversely, we assume that $T(u) \in 2\mathscr{E}_{sp}$. By Theorem 6.1, there is some $w \in X_{s,p}$ such that $T(w) = \dfrac{1}{2}T(u)$. Set $z := u/w^2$. Then $T(z) = 0$ (by (5.1)). By Theorem 6.1, items (4) to (6), we may write $z = e^{\imath\varphi}$, with

(1) $\varphi \in W^{s,p} \cap W^{1,sp}$ when $s \geq 1$;
(2) $\varphi \in W^{s,p} + W^{1,sp}$ when $0 < s < 1$.

We set $v := w\, e^{\imath\varphi/2}$; this v satisfies $v^2 = u$. When $1 < sp < 2$ or $s = p = 1$, Theorem 6.1, items (4) to (5) imply that $v \in X_{s,p}$. $\square$

[25] In the formula below, "deg" is the winding number.

Clearly, a positive answer to item (2) in Open Problem 6 would imply that, in the case $sp = 1$, a positive answer to $Q_{s,p,u}$ is equivalent to $T(u) \in 2\mathscr{E}_1$. Thus a weaker form of Open Problem 6 is

Open Problem 7. Assume that $0 < s < 1$ and $sp = 1$. Let $u \in X_{s,p}$ such that $T(u) \in 2\mathscr{E}_1$. Is it true that there is some $v \in X_{s,p}$ such that $v^2 = u$?
Equivalently, is it true that $\varphi \in Y^p \Longrightarrow \dfrac{\varphi}{2} \in Y^p$?
Equivalently, is it true that for each $u \in \overline{C^\infty(\mathbb{S}^n; \mathbb{S}^1)}^{W^{s,p}}$ there is some $v \in X_{s,p}$ such that $v^2 = u$?

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Université de Lyon ; Université Lyon 1 ; INSA de Lyon, F-69621 ; École Centrale de Lyon ; CNRS, UMR5208, Institut Camille Jordan, 43 blvd du 11 novembre 1918, F-69622 Villeurbanne-Cedex, France

E-mail address: `mironescu@math.univ-lyon1.fr`